A Simple Introduction to Linear Algebra

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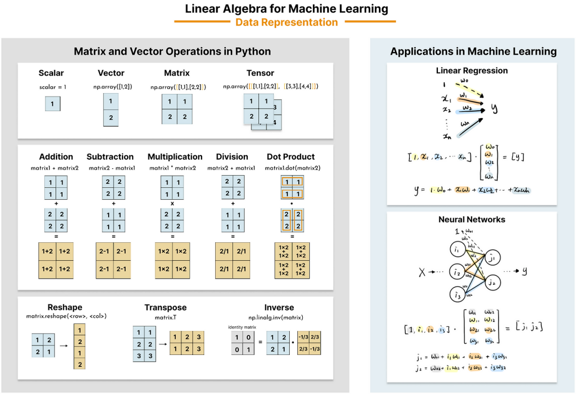
- 不要放过任何机会寻求专业的指导，不论是来自你的老师、同学还是你的 Hall Mentor
- 尽情做自己喜欢的事情，什么时候都不晚
- 关心自己，不要关心别人怎么看

What is Linear Algebra?

- The branch of mathematics concerning linear equations.
- 说人话: 关心线性等式及其变换
- 什么是线性等式: 单调即线性
- 什么是等式： $a = b$
- 什么是变换： Plus, Multiply, Transformation, Transpose ...

Linear Algebra 有什么用?

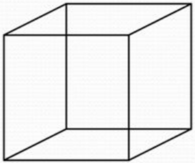
- Linear Algebra in **Machine/Deep Learning**



Linear Algebra 有什么用?

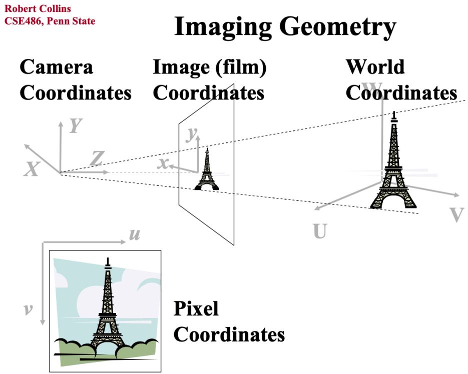
- Represent and manipulate 3D objects in computer graphics

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 \end{bmatrix}$$



Linear Algebra 有什么用?

- Computer Vision/Computer Graphics



Linear Algebra 有什么用?

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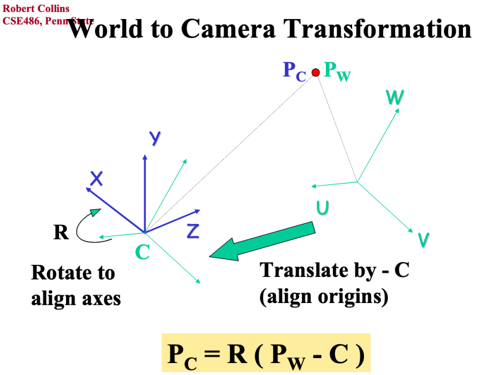


如何学好 Linear Algebra?

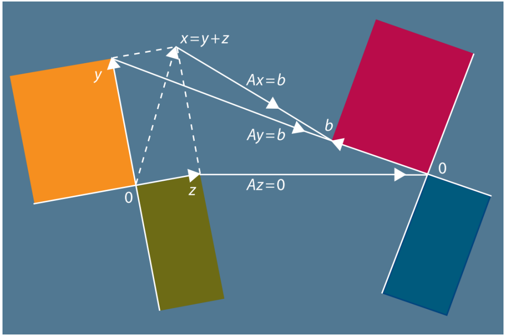
- 传奇经历： Midterm: Mean: 77 Mine: 65
- Final Grade: A
- 折算下来应该期末是快满分了

如何学好 Linear Algebra?

- Geometry thought: 有了 CV/CG 的启发，在学一些变换的时候应该要想象到在几何空间中的变换
- Basic Principles: 掌握最基本的运算/变换法则，不论是计算还是理解，都非常有用
- 学会发现关联

如何学好 Linear Algebra?

- Core of Linear Algebra: $A\mathbf{x} = \mathbf{b}$ and the Four Subspaces



Geometry Essense of Linear Algebra

- Vector
- The combination of vectors: Matrix
- Linear Space
- 3B1B Sec1

The geometry of linear equations

- $x\mathbf{c} + y\mathbf{d}$ is ***Linear Combination***. Relation: ***Column Vector***
- **Important Form:** $A\mathbf{x} = \mathbf{b}$
- Scalar · Vector
- Matrix Multiplication
- 3B1B Sec2
- 3B1B Sec3
- 3B1B Sec4

Exercise for Multiplication

- Scalar Product
- Matrix Product

How to solve a linear comnbination?

- Strict triangular form
- Elementary Row Operations
- Row-Echelon Form

Elementary Row Operations

- Interchange two rows
- Multiply a row by a nonzero real number
- Replace a row by the sum of that row and a multuple of a another row.
- 3B1B Sec5

Row-Echelon Form

Definition

A matrix is said to be in row echelon form if

- The first nonzero entry in each nonzero row is 1.
- If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
- If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Reduced Row-Echelon Form

Definition

A matrix is said to be in reduced row echelon form if

- The matrix is in row echelon form.
- The first nonzero entry in each row is the only nonzero entry in its c

A simple example of Gaussian Elimination

Problem

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

Other Matrix Operations

- Transpose：转置；行换到列，列换到行 $\mathbb{R}^{n\times m} \to \mathbb{R}^{m\times n}$

Algebra Principles

Theorem 1.4.1 *Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A + B) = \alpha A + \alpha B$

A lot of practice

- See the note

Consistency Theorem for Linear Systems

Definition

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be written as a linear combination of the column vectors of A.

Consistency Theorem for Linear Systems

Problem

Is this equation consistent?

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \end{aligned} \tag{1}$$

Introduction to Matrix Inversion

- Mathematics is interconnected
  - When we multiply a number by its **reciprocal** we get 1: $8 \times \frac{1}{8} = 1$
  - In the number theory, $AA^{-1} \equiv 1 (\mathrm{mod}\ p)$, which is called Multiplicative inverse (乘法逆元)
- Do matrices have similar properties?

Introduction to Matrix Inversion

- What is 1 in the linear algebra?
- $a \times 1 = a, 1 \times a = a, a \times a^{-1} = 1$

Introduction to Matrix Inversion

- Let $A \in \mathbb{R}^{n\times n}$, then
$$AA^{-1} = A^{-1}A = I$$
- 但是矩阵乘法有交换律吗?
- Example:

The matrices

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

are inverses of each other, since

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

How to solve the inverse matrix?

- Key: $AA^{-1} = I$, which is still a form of $A\mathbf{x} = \mathbf{b}$
- Method: Elementary Row Operations
- Example:

Compute A^{-1} if

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

What's more?

- If A is a square matrix, the most important question you can ask about it is whether it has an inverse A^{-1}. If it does, then $A^{-1}A = I$ and we say that A is **invertible or nonsingular**.

What's more?

- If A is singular —*i.e.*, A does not have an inverse —its determinant is 0 and we can find some non-zero vector x for which $Ax = 0$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A Real Life Example: Bus and Train

Description

A group took a trip on a bus, at \$3 per child and \$3.20 per adult for a total of 118.40. They took the train back at \$3.50 per child and \$3.60 per adult for a total of 135.20. How many children, and how many adults?

A Real Life Example: Bus and Train

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\begin{array}{c} \begin{matrix}\text{Child} & \text{Adult}\end{matrix} \\ \begin{bmatrix} x_1 & x_2 \end{bmatrix}\end{array} \begin{array}{c} \begin{matrix}\text{Bus} & \text{Train}\end{matrix} \\ \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}\end{array} = \begin{array}{c} \begin{matrix}\text{Bus} & \text{Train}\end{matrix} \\ \begin{bmatrix} 118.4 & 135.2 \end{bmatrix}\end{array}$$

This is just like the example above:

$$XA = B$$

The Determinant of a Matrix

- With each $n \times n$ matrix A, it is possible to associate a scalar, $\det A$, whose value will tell us whether the matrix is nonsingular. Before proceeding to the general definition, let us consider the following cases.

The Determinant of a Matrix

Case 1. 1 × 1 Matrices If $A = (a)$ is a 1×1 matrix, then A will have a multiplicative inverse if and only if $a \neq 0$. Thus, if we define

$$\det(A) = a$$

then A will be nonsingular if and only if $\det(A) \neq 0$.

Case 2. 2 × 2 Matrices Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

By Theorem 1.5.2, A will be nonsingular if and only if it is row equivalent to I. Then, if $a_{11} \neq 0$, we can test whether A is row equivalent to I by performing the following operations:

The Determinant of a Matrix

1. Multiply the second row of A by a_{11}:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix}$$

2. Subtract a_{21} times the first row from the new second row:

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

Since $a_{11} \neq 0$, the resulting matrix will be row equivalent to I if and only if

$$a_{11}a_{22} - a_{21}a_{12} \neq 0 \tag{1}$$

If $a_{11} = 0$, we can switch the two rows of A. The resulting matrix

$$\begin{bmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{bmatrix}$$

will be row equivalent to I if and only if $a_{21}a_{12} \neq 0$. This requirement is equivalent to condition (1) when $a_{11} = 0$. Thus, if A is any 2×2 matrix and we define

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

then A is nonsingular if and only if $\det(A) \neq 0$.

Implication: 1

- 刚刚的重要操作：Elementary Row Operations/Row Equivalant
- Why?

Compute with Minor and Cofactor

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix, and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the minor of a_{ij}. We define the cofactor A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

Than calculate the determinate with a ***cofactor expansion*** along one row or one column.

An example of computing determinate

Description

Calculate the determinate of the following matrix:

$$\begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

Implication: 2

- 为什么不像之前一样，先通过 Elementary Row Operations 再算 determinate 呢?
答案是肯定的!

Visualization for Determinate

- 3B1B Sec6

Properties of Determinates

- 不讲，因为这个只有你们计算的时候用得到
- 本课程是为了让大家欣赏 Linear Algebra，至少是不反感
- 但是为了让这个 Intro 更加完善，我还是会放出来

Properties of Determinates

In summation, if E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases} \tag{2}$$

Similar results hold for column operations. Indeed, if E is an elementary matrix, then E^T is also an elementary matrix (see Exercise 8 at the end of the section) and

$$\begin{aligned} \det(AE) &= \det\big((AE)^T\big) = \det\big(E^T A^T\big) \\ &= \det\big(E^T\big)\det\big(A^T\big) = \det(E)\det(A) \end{aligned}$$

Thus, the effects that row or column operations have on the value of the determinant can be summarized as follows:

I. Interchanging two rows (or columns) of a matrix changes the sign of the determinant.

II. Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.

III. Adding a multiple of one row (or column) to another does not change the value of the determinant.

Conclusion for Singularity

Conclusion

$A \in \mathbb{R}^{n \times m}$ singularity

- monsingular $\Leftrightarrow \exists A^{-1} \Leftrightarrow \vec{x} = A^{-1}\vec{b}$ (only unique solution)
  $\Leftrightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ (only trival solution)
  $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow \lambda \neq 0$.
- Singular : $N(A) \neq \{\vec{0}\}$; $\lambda = 0$